# Trace and Determinants

$A_{n\times n} \to$ square

$\mathrm{Tr}(A) = \sum_{i=1}^{n} a_{ii}$

Ex/ $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}_{3\times 3}$ $\quad \mathrm{Tr}(A) = 1+5+9 = 15$

## Determinants

$A_{n\times n}$ $\quad \det(A)$, $|A|$, $\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix}$

1x1 matrices : Ex/ $A = [5]_{1\times 1}$ $\quad \det(A) = 5$

2x2 matrices : $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ $\quad \det(A) = a_{11}\cdot a_{22} - a_{12}a_{21}$

Ex/ $A = \begin{bmatrix} 5 & 2 \\ 3 & 4 \end{bmatrix} \Rightarrow \det(A) = 5.4 - 2.3 = 14$

### nxn matrices $n \geqslant 3$ :

$A_{n\times n}$ 4x4

→ minor : The determinant of the matrix obtained by deleting $i^{th}$ row and $j^{th}$ column from the previous matrix. $(n-1)\times(n-1)$

→ $M_{ij}$

Ex/ $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$

$M_{23} = \begin{vmatrix} 1 & 2 \\ 7 & 8 \end{vmatrix} = 1.8 - 2.7 = -6$

$M_{31} = \begin{vmatrix} 2 & 3 \\ 5 & 6 \end{vmatrix} = 12 - 15 = -3$

signed (+ or −) minors

cofactor : $A_{ij} = (-1)^{i+j} M_{ij}$

$i+j$ = even ⇒ +
$i+j$ = odd ⇒ −

$A_{23} = (-1)^{2+3} M_{23} = -(-6) = 6$

$A_{31} = (-1)^{3+1} M_{31} = +(-3) = -3$

→ (right) cofactor expansion : $i^{th}$ row - cofactor expansion

$A = \begin{bmatrix} & & & \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ & & & \end{bmatrix}$ ($i^{th}$ row)

$$\sum_{j=1}^{n} a_{ij} A_{ij}$$

$i$ is fixed
$j$ runs from 1 to $n$

$= a_{i1}A_{i1} + a_{i2}A_{i2} + a_{i3}A_{i3} + \dots + a_{in}A_{in}$

entries of the row, cofactor

$= \boxed{\det(A)}$

OR

$j^{th}$ column cofactor expansion

$$\sum_{i=1}^{n} a_{ij} A_{ij}$$

$j$ is fixed
$i$ runs from 1 to $n$

$A = \begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{nj} \end{bmatrix}$ ($j^{th}$ column)

$= a_{1j}A_{1j} + a_{2j}A_{2j} + a_{3j}A_{3j} + \dots + a_{nj}A_{nj}$

$= \boxed{\det(A)}$

wrong cofactor expansion

$i^{th}$ row entries with another $k^{th}$ row cofactors

$a_{i1}A_{k1} + a_{i2}A_{k2} + \dots + a_{in}A_{kn} = \boxed{0}$

Warning! an easy way of 3x3 dets. ~~Sarrus Rule~~ not allowed.

Ex $A = \begin{bmatrix} 2 & 3 & 4 \\ -1 & -2 & 5 \\ -3 & 6 & 1 \end{bmatrix}_{3\times 3}$ $\quad \det(A) = ?$

→ 1st row co-factor expansion

$= a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}$

$= 2.(-1)^{1+1}M_{11} + 3.(-1)^{1+2}M_{12} + 4(-1)^{1+3}M_{13}$

$\begin{vmatrix} -2 & 5 \\ 6 & 1 \end{vmatrix} = -2-30 = -32$ $\quad \begin{vmatrix} -1 & 5 \\ -3 & 1 \end{vmatrix} = -1-(-15) = 14$ $\quad \begin{vmatrix} -1 & -2 \\ -3 & 6 \end{vmatrix} = -6-(6) = -12$

$= 2.(-32) + 3.(-1).14 + 4.(-12) = -154$

$-64 \quad -42 \quad -48$

4x4

$A = \begin{bmatrix} 2 & 3 & 4 \\ -1 & -2 & 5 \\ -3 & 6 & 1 \end{bmatrix}$ $\quad$ 3rd row cofactor expansion

$= a_{31}A_{31} + a_{32}A_{32} + a_{33}A_{33}$

$= (-3)(-1)^{3+1}M_{31} + 6.(-1)^{3+2}M_{32} + 1.(-1)^{3+3}M_{33}$

cofactor expansion

$$= (-3)(-1)^{3+1} M_{31} + 6 \cdot (-1)^{3+2} M_{32} + 1 \cdot (-1)^{3+3} M_{33}$$

$$M_{31} = \begin{vmatrix} 3 & 4 \\ -2 & 5 \end{vmatrix} = 15-(-8) = 23 \qquad M_{32} = \begin{vmatrix} 2 & 4 \\ -1 & 5 \end{vmatrix} = 10-(-4) = 14 \qquad M_{33} = \begin{vmatrix} 2 & 3 \\ -1 & -2 \end{vmatrix} = -4-(-3) = -1$$

det (A)

$$= -69 - 84 + (-1) = -154$$

wrong cofactor expansion example

1st row with 3rd row cofactors.

$$a_{11}A_{31} + a_{12}A_{32} + a_{13}A_{33} = 2 \cdot 23 + 3 \cdot -14 + 4 \cdot (-1) = 0$$

$$46 \quad -42 \quad -4$$

## Tricks and Properties of Determinants

* Choose the row/column with more zeroes for the cofactor expansion.

Ex/

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 5 & -2 & 0 \\ -3 & 4 & 0 \end{bmatrix}$$

3rd column cofactor exp.

$$\det(A) = \underset{3}{a_{13}} A_{13} + \underset{0}{a_{23}} A_{23} + \underset{0}{a_{33}} A_{33} = a_{13}(-1)^{1+3} M_{13} + 0 + 0$$

$$M_{13} = \begin{vmatrix} 5 & -2 \\ -3 & 4 \end{vmatrix} = 20-6 = 14$$

$$= 42$$

* If A has an all-zero row (column) then $\det(A) = 0$

Ex/

$$A = \begin{bmatrix} 1 & 2 & 3 & 0 & 43 \\ 99 & 5 & -1 & 0 & 32 \\ -1 & 2 & 3 & 0 & 6 \\ 3 & 1 & 5 & 0 & 90 \\ 150 & 27 & 16 & 0 & 56 \end{bmatrix}_{5\times5}$$

$$\det(A) = 0 + 0 + 0 + 0 + 0 = 0$$

4th column cofactor expansion

4x4 → 3x3 → 2x2

* Determinants of Diagonal / Lower Triangular / Upper Triangular Matrices $\Rightarrow \det(A) = \prod_{i=1}^{n} a_{ii}$

$$\begin{bmatrix} \# & 0 \\ 0 & \# \end{bmatrix} \qquad \begin{bmatrix} \# & 0 \\ \# & \# \end{bmatrix} \qquad \begin{bmatrix} \# & \# \\ 0 & \# \end{bmatrix}$$

Ex/

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -4 \end{bmatrix}_{3\times3}$$

$$\det(A) = 2 \cdot (-1)^{1+1} M_{11} + 0 + 0$$

1st row cofactor exp → $a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}$

$$M_{11} = \begin{vmatrix} 3 & 0 \\ 0 & -4 \end{vmatrix} = 3 \cdot (-4) - 0.$$

$$\det(A) = 2 \cdot 3 \cdot (-4) = -24$$

Ex/

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 96 & 3 & 0 \\ 54 & 26 & -4 \end{bmatrix}$$

$$\det(A) = 2 \cdot (-1)^{1+1} M_{11} + 0 + 0 = 2 \cdot 3 \cdot (-4) = -24$$

$$M_{11} = \begin{vmatrix} 3 & 0 \\ 26 & -4 \end{vmatrix} = 3 \cdot (-4) - 0 \cdot 26$$

$$A = \begin{bmatrix} 2 & 96 & 28 \\ 0 & 3 & 36 \\ 0 & 0 & -4 \end{bmatrix}$$

$$\det(A) = 2(-1)^{1+1} M_{11} + 0 + 0$$

1st column cofactor exp. → $a_{11}A_{11} + a_{21}A_{21} + a_{31}A_{31}$

$$M_{11} = \begin{vmatrix} 3 & 36 \\ 0 & -4 \end{vmatrix} = 3 \cdot (-4) - 0 \cdot 36$$

$$= 2 \cdot 3 \cdot (-4)$$

* $\det(A) = \det(A^T)$

* $\det(A) = \det(A^T)$

* $\det(AB) = \det(A)\det(B)$ !!!

Ex/ $A = \begin{bmatrix} 3 & 2 \\ 5 & 4 \end{bmatrix}$ $\quad B = \begin{bmatrix} -2 & -4 \\ 3 & 7 \end{bmatrix}$ $\quad AB = \underset{2\times2}{\begin{bmatrix} 3 & 2 \\ 5 & 4 \end{bmatrix}}\underset{2\times2}{\begin{bmatrix} -2 & -4 \\ 3 & 7 \end{bmatrix}} = \begin{bmatrix} 0 & 2 \\ 2 & 8 \end{bmatrix} \to \det = 0.8 - 2.2 = -4$

$-6+6 \quad -12+14$

$-10+12 \quad -20+28$

$\det(A) = 12 - 10 = 2$ $\qquad \det(B) = -14 - (-12) = -2$

$\det(A)\det(B) = 2 \cdot -2 = -4$

* Determinants of Elementary Matrices $\quad \left( I_n \to \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix} \to \det(I_n) = 1 \right)$

$\boxed{\begin{array}{l} 1^{st} \text{ Type} \to \det(E) = -1 \\ 2^{nd} \text{ Type} \to \det(E) = c \\ 3^{rd} \text{ Type} \to \det(E) = 1 \end{array}}$ $\quad [cr_i \to r_i] \quad c \neq 0$

$1^{st}$ Type: $I_n \xrightarrow{r_i \leftrightarrow r_j}$

Ex/ $E = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$ $r_2 \leftrightarrow r_4$ $\to \det(E) = 1 \cdot \underset{\downarrow\, T}{(-1)^{1+1}} \cdot \begin{vmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{vmatrix} + 0 + 0 + 0 = -1$

$\to 0 + 0 + 1 \cdot (-1)^{1+3} \underbrace{\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}}_{-1}$

$2^{nd}$ Type: $I_n \xrightarrow{cr_i \to r_i}$

Ex/ $E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix}$ $4r_3 \to r_3$ $\to \det(E) = 1 \cdot 1 \cdot 4 = 4$

diagonal.

3rd Type: $I_n \xrightarrow{cr_i + r_j \to r_j}$

Ex/ $E = \begin{bmatrix} 1 & 0 & 0 \\ 5 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $5r_1 + r_2 \to r_2$ $\to \det(E) = 1 \cdot 1 \cdot 1 = 1$

lower triangular

$E = \begin{bmatrix} 1 & 0 & -3 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \to \det(E) = 1 \cdot 1 \cdot 1 \cdot 1 = 1$

$I_4 \xrightarrow{-3r_3 + r_1 \to r_1}$

upper triangular

* $A_{n\times n} \xrightarrow[\text{1 row operation} \to E]{} A' = EA$ (left multip)

$A_{n\times n} \xrightarrow[\text{column operation} \to E]{} A'' = AE$ right multiplied

$\det(EA) = \det(E)\det(A)$ from !!!

Ex/ $A \xrightarrow[3r_1 \to r_1 \;(E_1)]{} E_1A \xrightarrow[-5r_1 + r_2 \to r_2 \;(E_2)]{} E_2(E_1A) \xrightarrow[r_2 \leftrightarrow r_3 \;(E_3)]{} E_3E_2E_1A \xrightarrow[-2r_3 \to r_3 \;(E_4)]{} A' = E_4E_3E_2E_1A$

If $\det(A) = 5$, what is $\det(A') = ?$

$\det(A') = \det(E_4E_3E_2E_1A) = \underbrace{\det(E_4)}_{-2}\underbrace{\det(E_3)}_{-1}\underbrace{\det(E_2)}_{1}\underbrace{\det(E_1)}_{3}\underbrace{\det(A)}_{5}$

$= 30$

Ex/ $A \xrightarrow[\cancel{-3r_1 + r_2 \to r_2} \;E_1]{} E_1A \xrightarrow[\cancel{-2r_2 + r_2} \;E_2]{} E_2E_1A \xrightarrow[\cancel{r_1 + r_3 \to r_3} \;E_3]{} \underbrace{\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{E_3E_2E_1A} \to \det = 0$ $\qquad \det(A) = ?$

$\det(E_3E_2E_1A) = 0 \to \det(E_3)\det(E_2)\det(E_1)\det(A) = 0$

$\neq 0$

$$\det(E_3E_2E_1A)=0 \rightarrow \underbrace{\det(E_3)}_{\neq 0}\,\underbrace{\det(E_2)}_{\neq 0}\,\underbrace{\det(E_1)}_{\neq 0}\,\underbrace{\det(A)}=0$$

$-1, 1, c\neq 0$

$$\Rightarrow \det(A)=0$$

Ex

$$A=\begin{bmatrix} 5 & 7 & 8 & -14 & 0 \\ 3 & -2 & 6 & 4 & -3 \\ 5 & 1 & 9 & -2 & 6 \\ -5 & 4 & 0 & -8 & -7 \\ 96 & 5 & -3 & -10 & 56 \end{bmatrix} \qquad \det(A)=?$$

$5\times 5$

$$\xrightarrow[E]{2c_2+c_4\to c_4} \underbrace{\begin{bmatrix} 5 & 7 & 8 & 0 & 0 \\ 3 & -2 & 6 & 0 & -3 \\ 5 & 1 & 9 & 0 & 6 \\ -5 & 4 & 0 & 0 & -7 \\ 96 & 5 & -3 & 0 & 56 \end{bmatrix}}_{\det=0}$$

$AE$

$$\det(A)\underbrace{\det(E)}_{\neq 0}=0 \Rightarrow \det(A)=0$$

Ex

$$A=\begin{bmatrix} 2 & 3 \\ -5 & 4 \end{bmatrix} \xrightarrow[\text{2nd type } E_1]{\frac{1}{2}r_1\to r_1} \underbrace{\begin{bmatrix} 1 & 3/2 \\ -5 & 4 \end{bmatrix}}_{\det = 4-(-\frac{15}{2}) = \frac{23}{2}} = A' = EA$$

$$\det(A')=\underbrace{\det(E)}_{1/2}\,\underbrace{\det(A)}_{23}=\frac{23}{2}$$

$$\det(A)=8-(-15)=23$$

$$A' \xrightarrow[E_2]{5r_1+r_2\to r_2} \begin{bmatrix} 1 & 3/2 \\ 0 & \frac{23}{2} \end{bmatrix} = A'' = E_2A'$$

$\frac{15}{2}+4$ 3rd type

$$\det(A'')=\underbrace{\det(E_2)}_{1}\,\underbrace{\det(A')}_{\frac{23}{2}}=\frac{23}{2}$$